Econ Theory Bull (2015) 3:169–180
DOI 10.1007/s40505-014-0039-3

RESEARCH ARTICLE

On social welfare functions on infinite utility streams satisfying Hammond Equity and Weak Pareto axioms: a complete characterization

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Received: 21 February 2014 / Accepted: 4 April 2014 / Published online: 29 April 2014

Abstract This paper examines the problem of aggregating infinite utility streams with a social welfare function that respects the Hammond Equity and Weak Pareto axioms. The paper provides a complete characterization of domains (of the one period utilities) on which such an aggregation is possible. A social welfare function satisfying the Hammond Equity and Weak Pareto axioms exists on precisely those domains which are well-ordered sets in which the elements of the set are ordered according to the decreasing magnitude of the numbers belonging to the set. We show through examples how this characterization can be applied to obtain a number of results in the literature, as well as some new ones.

Keywords Domain restrictions · Hammond Equity axiom · Infinite utility streams · Order types · Social welfare function · Weak Pareto axiom · Well-ordered set.

Journal of Economic Literature Classification Numbers D60 · D70 · D90.

1 Introduction

In this paper, we examine the *extent* to which intergenerational equity principles come into conflict with principles of efficiency, when one insists on obtaining *social welfare functions* on the space of infinite utility streams which respect both principles. The

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only way to do this convincingly is to provide *complete* characterizations of the space of infinite utility streams on which such social welfare functions are to be defined.

The equity principle that we are concerned with is Hammond Equity, which belongs to the class of *consequentialist* equity concepts, dealing with situations in which the *distribution* of utilities of generations changes in specific ways. It is one of the key consequentialist equity concepts, the other being the Pigou-Dalton transfer principle.[1] It was introduced by Hammond (1976), who called it the *Equity axiom*, and is in the spirit of the *Weak Equity axiom* of Sen (1973).

Turning to the efficiency principle, it is known [see Alcantud and Garcia-Sanz (2013)] that there is no social welfare function satisfying Hammond Equity and Strong Pareto, if the domain set (Y) consists of at least four distinct elements.[2] That is, an impossibility result arises as soon as we admit a situation in which Hammond Equity can play a role in ranking two utility streams. On the other hand, if we consider the efficiency principle of Monotonicity, the combination of Hammond Equity and Monotonicity would clearly be satisfied by the trivial social welfare function which assigns the same welfare number to *all* utility streams. We choose a middle ground and focus on the efficiency principle of *Weak Pareto*.[3]

One can justifiably take the position that the Weak Pareto axiom is more compelling than the Strong Pareto axiom in the context of evaluating infinite utility streams. It requires that society should consider one stream of well-being to be superior to another if *every* generation is better off in the former compared to the latter. It is debatable whether in comparing two utility streams, society is always better off if one generation is (or a finite number of generations are) better off and all other generations are unaffected, so the Strong Pareto axiom might not be self-evident.

The objective of our paper, then, can be summarized as follows. We would like to completely characterize the domain sets Y for which there exist social welfare functions on the space of utility streams $X = Y^{\mathbb{N}}$, satisfying the Hammond Equity and Weak Pareto axioms.[4]

[1] Hammond Equity has several variations which have been discussed in the literature. Strong Equity [see d'Aspremont and Gevers (1977), and Dubey and Mitra (2014)] and Hammond Equity for the Future [see Asheim et al. (2007) and Banerjee (2006)] are notable variations. Weak Hammond Equity, introduced in Bosmans and Ooghe (2013) in a framework with a finite number of individuals, is conceptually similar to Hammond Equity for the Future, which has been analyzed in the infinite horizon inter-temporal context. The Pigou-Dalton transfer principle has been discussed in Hara et al. (2008); Altruistic Equity, a variation of the Pigou-Dalton transfer principle, has been discussed by Sakamoto (2012).

[2] We use the standard framework in which the space of infinite utility streams is given by $X = Y^{\mathbb{N}}$, where Y is a nonempty set of real numbers, and $\mathbb{N}$ is the set of natural numbers. There is a considerable literature on equitable and efficient social welfare quasi-orderings in a framework in which there are a finite number of individuals (generations). For this literature, as well as a recent complete characterization of the maximin social welfare quasi-ordering, see Bosmans and Ooghe (2013).

[3] The various efficiency concepts discussed here, as well as the concept of Hammond Equity, are defined precisely in Sect. 2.3.

[4] It should be noted that there are alternate specifications of X for which the objective may be addressed. For example, X could be l_{∞}^{+}, which does not have the product structure used in our paper. Our choice is dictated by two considerations. First, since much of the standard literature on this topic uses this product structure, it becomes straightforward to relate our results to those in this literature. Second, because of the product structure, it is possible to obtain the characterization in terms of the structure of the set Y (instead of the set X), and this makes the characterization result relatively easy to apply.

In pursuing this objective, we build, of course, on results on this theme already available in the literature. Alcantud and Garcia-Sanz (2013) have constructed a social welfare function satisfying the Hammond Equity and Weak Pareto axioms when Y is the set of natural numbers $\mathbb{N}$. On the other hand, when $Y = [0, 1]$, Alcantud (2012) has shown that there is no social welfare function satisfying the Hammond Equity and Weak Pareto axioms. These results naturally lead one to investigate whether it is the countability of the set Y that is crucial in allowing possibility results to emerge.

This turns out to be *not* the case. Our complete characterization result (Theorem 1) establishes that the domains, Y, for which there exists a social welfare function satisfying the Hammond Equity and Weak Pareto axioms are precisely those which are well-ordered sets, with the elements of the set being ordered according to the decreasing magnitude ($<$) of the numbers belonging to the set.[5] In particular, our characterization reproduces the possibility result of Alcantud and Garcia-Sanz (2013) that when Y is the set of positive integers, there is a social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the Hammond Equity and Weak Pareto axioms (since $Y(<)$ is well-ordered). But, it also produces the somewhat surprising new result that when Y is the set of negative integers, there is no social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the Hammond Equity and Weak Pareto axioms (since $Y(<)$ is not well-ordered).

2 Preliminaries

2.1 Notation

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{M}$ be the sets of real numbers, natural numbers $\{1, 2, 3, \cdots\}$, and negative integers respectively. For all $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.2 Strictly ordered sets, order types and well-ordered sets

We recall a few concepts from the mathematical literature dealing with *strictly ordered* sets, *order types* and *well-ordered* sets.

We will say that the set A is *strictly ordered* by a binary relation $\Re$ if $\Re$ is *connected* (if $a, a' \in A$ and $a \neq a'$, then either $a\Re a'$ or $a'\Re a$ holds), *transitive* (if $a, a', a'' \in A$ and $a\Re a'$ and $a'\Re a''$ hold, then $a\Re a''$ holds) and *irreflexive* ($a\Re a$ holds for no $a \in A$). In this case, the strictly ordered set will be denoted by $A(\Re)$.

For example, the set $\mathbb{N}$ is strictly ordered by the binary relation $<$ (where $<$ denotes the usual "less than" relation on the reals); thus $\mathbb{N}(<)$ is a strictly ordered set. Similarly, $\mathbb{M}(<)$ is a strictly ordered set. It can be easily verified that $\mathbb{N}(>)$, $\mathbb{M}(>)$ are also strictly ordered sets (where $>$ denotes the usual "greater than" relation on the reals).

We will say that a strictly ordered set $A'(\Re')$ is *similar* to the strictly ordered set $A(\Re)$ if there is a one-to-one function f mapping A onto A', such that:

[5] Well-ordered sets and the order types are defined precisely in Sect. 2.2.

$$a_1, a_2 \in A \quad \text{and} \quad a_1 \Re a_2 \quad \Longrightarrow f(a_1) \Re' f(a_2).$$

The function $f : \mathbb{N} \to \mathbb{M}$ given by $f(n) = -n$ for all $n \in \mathbb{N}$ is a one-to-one function mapping $\mathbb{N}$ onto $\mathbb{M}$. Furthermore, whenever $a_1, a_2 \in \mathbb{N}$, and $a_1 < a_2$, we have $f(a_1) = -a_1 > -a_2 = f(a_2)$. Thus, the strictly ordered set $\mathbb{M}(>)$ is *similar* to the strictly ordered set $\mathbb{N}(<)$.

However, it is worth noting that $\mathbb{M}(<)$ is *not* similar to $\mathbb{N}(<)$. For, if it were, there would be a one-to-one function f mapping $\mathbb{N}$ onto $\mathbb{M}$, such that whenever $a_1, a_2 \in \mathbb{N}$, and $a_1 < a_2$, we have $f(a_1) < f(a_2)$. Then, denoting $f(1)$ by z_1, we note that $z_1 \in \mathbb{M}$ and we can find $\bar{z} \in \mathbb{M}$ such that $\bar{z} < z_1$. Since f is a one-to-one map of $\mathbb{N}$ onto $\mathbb{M}$, we can find $\bar{n} \in \mathbb{N}$, $\bar{n} \neq 1$, such that $f(\bar{n}) = \bar{z}$. Then $1 < \bar{n}$, and we have $z_1 = f(1) < f(\bar{n}) = \bar{z}$, a contradiction.

We now specialize to strictly ordered subsets of the reals. With S a non-empty subset of $\mathbb{R}$, let us define[6] two *order types* as follows. We will say that the strictly ordered set $S(<)$ is:

1. Of order type ω if $S(<)$ is similar to $\mathbb{N}(<)$;
2. Of order type ω^* if $S(<)$ is similar to $\mathbb{M}(<)$.

An element $s_0 \in S$ is called a *first element* of $S(<)$ if $s < s_0$ holds for no $s \in S$. A strictly ordered set $S(<)$ is said to be *well-ordered* if each non-empty subset of S has a first element. It follows from these definitions that if the strictly ordered sets $S(<)$ and $T(<)$ are similar, then $S(<)$ is well-ordered if and only if $T(<)$ is well-ordered; see [Ciesielski (1997, Proposition 4.1.4, p.39)].

The following lemma is the basic characterization result on well-ordered sets, and it brings together the various concepts introduced above; see [Sierpinski (1965, Theorem 1, p. 262), Jech (1973, Proposition, p. 23), and Ciesielski (1997, Theorem 4.3.2, p. 51)].

Lemma 1 *Let S be a non-empty subset of* $\mathbb{R}$ *and* $S(<)$ *a strictly ordered set. The necessary and sufficient condition for S(<) to be a well-ordered set is that it should contain no subset of order type* ω^**; that is, it should contain no infinite strictly decreasing sequence.*

2.3 Social welfare function, efficiency and equity

Let Y, a non-empty subset of $\mathbb{R}$, be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^{\mathbb{N}}$ is the set of all possible utility streams. We denote an element of X by x, or alternately by $\langle x_n \rangle$, depending on the context. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \ldots)$, where, for all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that the generation of period n earns.

[6] The names "order type ω" and "order type ω^*" are discussed and characterized in Sierpinski (1965, p. 210, 226).

A *social welfare function* (SWF) is a mapping $W : X \to \mathbb{R}$.[7] The following equity and efficiency properties are considered to be desirable attributes of a social welfare function, and are discussed in this paper.

Definition 1 Hammond Equity (HE): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$, while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $W(x) \geq W(y)$.

Definition 2 Weak Pareto (WP): For $x, y \in X$ if $x \gg y$, then $W(x) > W(y)$.

Definition 3 Strong Pareto (SP): For $x, y \in X$, if $x > y$, then $W(x) > W(y)$.

Definition 4 Monotonicity (M): For $x, y \in X$, if $x \geq y$, then $W(x) \geq W(y)$.

Remark 1 These equity and efficiency properties are invariant to monotone increasing transformations of the units of measurement of individual utilities. We can spell out the content of this observation as follows: Suppose $\mathbf{Y}$ is a non-empty subset of $\mathbb{R}$, and $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$, and $\mathbf{W}$ is a social welfare function satisfying any of the above equity and efficiency properties on $\mathbf{X}$. Let Y be a non-empty subset of $\mathbb{R}$, and g be any monotone increasing function from Y to $\mathbf{Y}$ and $X = Y^{\mathbb{N}}$. Then $W(x) = \mathbf{W}(g(x_1), g(x_2), \ldots)$ is a social welfare function satisfying the corresponding property on X.

3 The characterization result

In this section we present our complete characterization result, which can be stated as follows:

Theorem 1 *Let Y be a non-empty subset of $\mathbb{R}$. There exists a social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Hammond Equity and Weak Pareto axioms if and only if $Y(<)$ is a well-ordered set. When $Y(<)$ is well-ordered, it is possible to explicitly construct such a social welfare function.*

We establish the possibility part of the Theorem in Sect. 3.1 (Proposition 1), and the impossibility part of the Theorem in Sect. 3.2 (Proposition 2).

Compared to the partial results available in the literature, our characterization is complete. Further, we see that it is easily applicable as the following examples illustrate.

Example 1 Let Y be a non-empty subset of $\mathbb{N}$. Since any non-empty subset of $\mathbb{N}(<)$ has a first element [see Munkres (1975, Theorem 4.1, p. 32)], $Y(<)$ is well-ordered. Using Theorem 1, there is a function $W : X \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms. This provides an alternative approach to the possibility result noted in Alcantud and Garcia-Sanz (2013) for $Y = \mathbb{N}$.

[7] In the literature, it is more common to start with a binary relations on X, denoted by $\succsim$, with symmetric and asymmetric parts denoted by $\sim$ and $\succ$ respectively, defined in the usual way. A *social welfare order* (SWO) is defined to be a complete and transitive binary relation. Given a SWO $\succsim$ on X, we say that $\succsim$ can be *represented* by a real-valued function if there is a mapping $W : X \to \mathbb{R}$ such that for all $x, y \in X$, we have $x \succsim y$ if and only if $W(x) \geq W(y)$. Thus, a social welfare function is obtained as a representation of a SWO. In our paper, we treat the social welfare function itself as the primitive concept. Our results, of course, can be rephrased in terms of representable social welfare orders.

Example 2 Let Y be defined by:

$$Y = \left\{ \frac{n}{n+1} \right\}_{n \in \mathbb{N}}$$

Define $f : \mathbb{N} \to Y$ by $f(n) = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. Then, f is a one-to-one function mapping $\mathbb{N}$ onto Y. Further, if $n, n' \in \mathbb{N}$ and $n < n'$, then $f(n) = \frac{n}{n+1} < \frac{n'}{n'+1} = f(n')$. Thus, $Y(<)$ is similar to $\mathbb{N}(<)$. Since $\mathbb{N}(<)$ is well-ordered (see Example 1), so is $Y(<)$. Using Theorem 1, there is a function $W : X \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms.

Example 3 Let $Y = \mathbb{M}$. Then $Y(<)$ is of order type ω^* and is therefore not a well-ordered set by Lemma 1. Using Theorem 1, there is no function $W : X \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms.

Example 4 Let Y be defined by:

$$Y = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}}$$

Then, $Y(<)$ is not a well-ordered set by Lemma 1. Using Theorem 1, there is no function $W : X \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms.

Example 5 Let $Y = [0, 1]$. Then, $Y(<)$ is not a well-ordered set by Lemma 1. Using Theorem 1, there is no function $W : X \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms. This provides an alternative approach to the impossibility result noted in Alcantud (2012).

3.1 The possibility result

We start with the possibility part of the result in Theorem 1. Our possibility result generalizes the corresponding result of Alcantud and Garcia-Sanz (2013), who established it for the domain $Y = \mathbb{N}$, as is clear from Examples 1 and 2 above. We use the same welfare function,[8] that they do, showing that this function suffices under the more general condition that $Y(<)$ is a well-ordered set.

The fact that one can explicitly write down the social welfare function is a bonus of our possibility result. In addition, the social welfare function has the desirable property that it satisfies the Monotonicity axiom.

Proposition 1 *Let Y be a non-empty subset of $\mathbb{R}$, and suppose that $Y(<)$ is well-ordered. For $x = (x_n)_{n=1}^{\infty} \in X \equiv Y^{\mathbb{N}}$, the function $W : X \to \mathbb{R}$, given by:*

$$W(x) = \min\{x_n\}_{n \in \mathbb{N}}$$

[8] This is the same welfare function as the one used by Basu and Mitra (2007), in establishing a possibility result satisfying the Anonymity and Weak Pareto axioms. For a social welfare function $W : X \to \mathbb{R}$, the Anonymity axiom can be stated as follows. For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for every $k \in \mathbb{N} \sim \{i, j\}$, $x_k = y_k$, then $W(x) = W(y)$.

is well-defined, and satisfies the Hammond Equity and Weak Pareto axioms.

Proof For any $x \in X$, the set $F(x) = \{r : r \in \mathbb{R},\ r = x_n \text{ for some } n \in \mathbb{N}\}$ is a non-empty subset of Y. Since $Y(<)$ is well-ordered, we can infer that $F(x) \subset Y$ must contain a first element, and so:

$$W(x) = \min\{x_n\}_{n \in \mathbb{N}}$$

is well-defined.

To verify that W satisfies Hammond equity, let $x, y \in X$, with $x_i < y_i < y_j < x_j$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$. There are two possibilities to consider: (i) $x_i = W(x)$; (ii) $x_i > W(x)$.

In case (i), $y_j > y_i > W(x)$, and for all $k \in \mathbb{N} \setminus \{i, j\}$, $y_k = x_k \geq W(x)$. Thus, $W(y) \geq W(x)$.

In case (ii), $x_j > x_i > W(x)$, and so there is $m \in \mathbb{N} \setminus \{i, j\}$, such that $x_m = W(x)$. Then, we have $y_j > y_i > x_i > W(x)$, and for all $k \in \mathbb{N} \setminus \{i, j\}$, $y_k = x_k \geq x_m = W(x)$. Thus, $W(y) \geq W(x)$.

To verify that W satisfies Weak Pareto, let $x, y \in X$ with $y \gg x$. Then, $W(y) = y_m$ for some $m \in \mathbb{N}$, and $y_m > x_m \geq W(x)$, so that $W(y) > W(x)$. □

3.2 The impossibility result

We now turn to the impossibility part of the complete characterization result stated in Theorem 1. We closely follow the technique of proof used in Dubey and Mitra (2011). The novelty is in showing that the Hammond Equity axiom can play a role in the current context similar to that played by the procedural equity notion of Anonymity in our earlier paper.

Proposition 2 *Let* $\mathbf{Y}$ *be a non-empty subset of* $\mathbb{R}$ *such that* $\mathbf{Y}(<)$ *is not well-ordered. Then there is no social welfare function* $\mathbf{W} : \mathbf{X} \to \mathbb{R}$ *satisfying the Hammond Equity and Weak Pareto axioms (where* $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$*).*

Proof Suppose on the contrary that there is a social welfare function $\mathbf{W} : \mathbf{X} \to \mathbb{R}$ satisfying the Hammond Equity and Weak Pareto axioms (where $\mathbf{X} = \mathbf{Y}^{\mathbb{N}}$). Since $\mathbf{Y}(<)$ is not well-ordered, we can use Lemma 1 to infer that $\mathbf{Y}$ contains a non-empty subset $\mathbf{Y}'$ such that $\mathbf{Y}'(<)$ is of order type ω^*. That is, there is a one-to-one mapping, g, from $\mathbb{M}$ onto $\mathbf{Y}'$ such that:

$$a_1, a_2 \in \mathbb{M} \quad \text{and} \quad a_1 < a_2 \Longrightarrow g(a_1) < g(a_2).$$

Thus, g is an increasing function from $\mathbb{M}$ to $\mathbf{Y}'$. Define $Y = \mathbb{M}$ and $X = Y^{\mathbb{N}}$, and $W : X \to \mathbb{R}$ by $W(x) = \mathbf{W}(g(x_1), g(x_2), \ldots)$. Note that $g(x_n) \in \mathbf{Y}' \subset \mathbf{Y}$ for each $n \in \mathbb{N}$, so W is well-defined. By Remark 1 (Sect. 2.3), W satisfies the Hammond Equity and Weak Pareto axioms on X. The proof proceeds now to show that this leads us to a contradiction.

Let Q be a fixed enumeration of the rationals in $(0, 1)$. Then, we can write:

$$Q = \{q_1, q_2, q_3, \ldots\}.$$

For any real number $t \in (0, 1)$, there are infinitely many rational numbers from Q in $(0, t)$ and in $[t, 1)$. For each real number $t \in (0, 1)$, we can then define the set $M(t) = \{n \in \mathbb{N} : q_n \in (0, t)\}$ and the sequence $\langle m_p(t)\rangle$ as follows:

$$m_1(t) = \min\{n \in \mathbb{N} : q_n \in (0, t)\};$$

and for $p \in \mathbb{N}$, $p > 1$,

$$m_p(t) = \min\{n \in \mathbb{N} \sim \{m_1(t), \ldots, m_{p-1}(t)\} : q_n \in (0, t)\}.$$

The sequence $\langle m_p(t)\rangle$ is then well-defined, and:

$$m_1(t) < m_2(t) < m_3(t) \ldots;$$

and $M(t) = \{m_1(t), m_2(t), \ldots\}$. We break up the proof into the following steps.

Step 1 [Defining the sequences $\langle x(t)\rangle$ and $\langle z(t)\rangle$ and intervals for distinct real numbers in (0, 1)]
For each real number $t \in (0, 1)$, we note that $M(t)$ contains infinitely many elements. Then, we can define sequences $\langle x(t)\rangle$ and $\langle z(t)\rangle$ by:

$$\left.\begin{array}{lll} \text{(i)} & x_n(t) = -4m_n(t) & \text{for all } n \in \mathbb{N} \\ \text{(ii)} & z_n(t) = x_n(t) + 1 & \text{for all } n \in \mathbb{N}. \end{array}\right\} \tag{1}$$

Note that the sequence $\langle x_n(t)\rangle$ will contain a subset of the negative even integers in decreasing order of magnitude with n. Also, the sequence $\langle z_n(t)\rangle$ will contain a subset of the negative odd integers in decreasing order of magnitude with n. By the Weak Pareto axiom, $W(\langle z_n(t)\rangle) > W(\langle x_n(t)\rangle)$. Thus, for each $t \in (0, 1)$, $I(t) \equiv [W(\langle x_n(t)\rangle), W(\langle z_n(t)\rangle)]$ is a non-degenerate closed interval in $\mathbb{R}$.

Step 2 [Comparing $\langle x(t)\rangle$ with $\langle x(s)\rangle$]
Let t, s be arbitrary real numbers in $(0, 1)$, with $t < s$. Note that if $n \in M(t)$, then $n \in M(s)$. Since there are an infinite number of rationals from Q in $[t, s)$, there will be an infinite number of distinct elements of $\mathbb{N}$ in $L(t, s) \equiv M(s) \setminus M(t) = \{n \in \mathbb{N} : q_n \in [t, s)\}$. For any $n \in L(t, s)$, we have $n = m_k(s) < m_k(t)$ for some k. That is, by (1) (i), for each $n \in L(t, s)$ it must be the case that $x_n(t) < x_n(s) < 0$. Consequently, one has:

$$x_n(s) \geq x_n(t) \text{ for all } n \in \mathbb{N}. \tag{2}$$

One can strengthen the conclusion in (2) as follows. Define:

$$\left.\begin{array}{ll} \text{(i)} & N \equiv \min\{n \in \mathbb{N} : n \in L(t, s)\} \\ \text{(ii)} & K \in \mathbb{N} : m_K(s) = N. \end{array}\right\} \tag{3}$$

For $k < K$, observe that $m_k(s) = m_k(t)$. By (1)(i), $x_k(t) = -4m_k(t) = -4m_k(s) = x_k(s)$.
For $k = K$, observe that $m_K(s) = N < m_K(t)$. By (1)(i), $x_K(t) = -4m_K(t) \leq -4(N+1) = -4m_K(s) - 4 = x_K(s) - 4$, and so $x_K(s) - x_K(t) \geq 4$.
For $k > K$, since an additional element of $M(s)$ has been used up for $N = m_K(s)$, compared with $M(t)$, we must have $m_k(s) < m_k(t)$. Thus, by (1) (i), $x_k(s) - x_k(t) \geq 4$. To summarize, we have:

$$\left.\begin{array}{ll}\text{(i)} & x_k(s) - x_k(t) \geq 4 \ \text{ for all } k \geq K \\ \text{(ii)} & x_k(s) = x_k(t) \text{ for all } k < K \text{ (if any).}\end{array}\right\} \tag{4}$$

Step 3 [Comparing $z(t)$ with $x(s)$]
Let t, s be arbitrary real numbers in (0, 1), with $t < s$. There are two possibilities to consider: (a) $K = 1$; (b) $K \geq 2$, where K is defined by (3) (ii).
(a) $K = 1$. In this case, by (4),

$$x_n(s) - x_n(t) \geq 4 \quad \text{for all} \ \ n \in \mathbb{N}.$$

(b) $K \geq 2$. Define the sequence $\langle y_n^1 \rangle$ as follows:

$$y_n^1 = \begin{cases} x_n(s) - 1 & \text{if } n \in \{1, K\}; \\ z_n(t) & \text{otherwise.} \end{cases}$$

We can use (1) (ii) and (4) (i) to obtain:

$$y_K^1 = x_K(s) - 1 \geq x_K(t) + 3 > x_K(t) + 1 = z_K(t) \tag{5}$$

Also, using (1) (ii) and (4) (ii), we get:

$$y_1^1 = x_1(s) - 1 = x_1(t) - 1 = z_1(t) - 2 < z_1(t) \tag{6}$$

Combining (5) and (6), and using the fact that $y_1^1 = x_1(s) - 1 > x_K(s) - 1 = y_K^1$, we obtain:

$$z_1(t) > y_1^1 > y_K^1 > z_K(t). \tag{7}$$

Since, $y_n^1 = z_n(t)$ for all $n \in \mathbb{N} \setminus \{1, K\}$, and (7) holds, we can apply Hammond Equity to infer that:

$$W(\langle y_n^1 \rangle) \geq W(\langle z_n(t) \rangle). \tag{8}$$

Next, we define sequences $\langle y_n^p \rangle$ for $p \in \{2, \ldots, K-1\}$ recursively as follows:

$$y_n^p = \begin{cases} x_n(s) - 1 & \text{if } n \in \{p, K+p-1\}; \\ y_n^{p-1} & \text{otherwise.} \end{cases}$$

We can now essentially repeat the arguments leading to (5)–(8) as follows. We can use (1) (ii) and (4) (i) to obtain:

$$y^{p}_{K+p-1} = x_{K+p-1}(s) - 1 \geq x_{K+p-1}(t) + 3 > x_{K+p-1}(t) + 1 = z_{K+p-1}(t) = y^{p-1}_{K+p-1}. \tag{9}$$

Also, using (1) (ii) and (4) (ii), we get:

$$y^{p}_{p} = x_{p}(s) - 1 = x_{p}(t) - 1 = z_{p}(t) - 2 < z_{p}(t) = y^{p-1}_{p} \tag{10}$$

Combining (9) and (10) and using the fact that $y^{p}_{p} = x_{p}(s) - 1 > x_{K+p-1}(s) - 1 = y^{p}_{K+p-1}$, we obtain:

$$y^{p-1}_{p} > y^{p}_{p} > y^{p}_{K+p-1} > y^{p-1}_{K+p-1}. \tag{11}$$

Since, $y^{p}_{n} = y^{p-1}_{n}$ for all $n \in \mathbb{N} \setminus \{p, K+p-1\}$ and (11) holds, we can apply the Hammond Equity axiom to infer that:

$$W(\langle y^{p}_{n}\rangle) \geq W(\langle y^{p-1}_{n}\rangle). \tag{12}$$

Combining (8) and (12) for $p \in \{2, \ldots, K-1\}$, we get:

$$W(\langle y^{K-1}_{n}\rangle) \geq W(\langle y^{K-2}_{n}\rangle) \ldots \geq W(\langle y^{1}_{n}\rangle) \geq W(\langle z_{n}(t)\rangle). \tag{13}$$

Observe that by construction, denoting the set $\{1, 2, \ldots, K-1; K, \ldots, 2(K-1)\}$ by $J(t, s)$,

$$\left.\begin{array}{lll} (i) & x_i(s) = y^{K-1}_{i} + 1 > y^{K-1}_{i} & \text{for all } i \in J(t,s), \\ (ii) & x_i(s) \geq x_i(t) + 4 = z_i(t) + 3 = y^{K-1}_{i} + 3 > y^{K-1}_{i} & \text{otherwise.} \end{array}\right\} \tag{14}$$

Using (14) and the Weak Pareto axiom,

$$W(\langle x_n(s)\rangle) > W(\langle y^{K-1}_{n}\rangle). \tag{15}$$

Combining (14) and (15), we get

$$W(\langle x_n(s)\rangle) > W(\langle z_n(t)\rangle). \tag{16}$$

Step 4 Let t, s be arbitrary real numbers in $(0, 1)$, with $t < s$. Then, by (16), the interval $I(s)$ lies entirely to the right of the interval $I(t)$ on the real line. That is, for arbitrary real numbers t, s in $(0, 1)$, with $t \neq s$, the intervals $I(t)$ and $I(s)$ are disjoint. Thus, we have a one-to-one correspondence between the real numbers in $(0, 1)$ (which is an uncountable set) and a set of non-degenerate, pairwise disjoint closed intervals of the real line (which is countable). This contradiction establishes the Proposition.

4 Concluding remarks

We now have two complete characterization results of domains Y for which there exist social welfare functions satisfying a consequential equity axiom along with an efficiency axiom. The first of these [see the Theorem in Dubey and Mitra (2014)] deals with the consequential equity axiom known as *Strong Equity* and the efficiency axiom of Monotonicity [9]. It states that there exists a social welfare function that combines the Strong Equity axiom and Monotonicity if and only if the cardinality of Y is at most five.

The second (contained in Theorem 1 of the current paper) deals with the equity axiom of Hammond and the efficiency axiom of Weak Pareto. Weak Pareto is not directly comparable to Monotonicity. However, we may note, for the purpose of our discussion here that Theorem 1 would remain valid if in its statement Weak Pareto was replaced by *Diamond's version of Weak Pareto* (abbreviated henceforth as DWP) which demands both Weak Pareto and Monotonicity.[10] Then, the characterization result of Theorem 1 would involve a *weaker* equity restriction but a *stronger* efficiency restriction compared to the Theorem in Dubey and Mitra (2014).

In general the two axiom *sets*, Strong Equity plus Monotonicity in Dubey and Mitra (2014), and Hammond Equity plus DWP in the current paper, are not comparable. However, the complete characterization results that we have provided make them comparable in a particularly simple way.

We note that for domain sets Y on which there exist social welfare functions satisfying Strong Equity plus Monotonicity, the cardinality of Y is at most five, and so Y is a *finite set*. Thus, $Y(<)$ is well-ordered, so by Theorem 1, there also exist social welfare functions satisfying Hammond Equity plus DWP. On the other hand, for the domain set $Y = \mathbb{N}$, $Y(<)$ is well-ordered and so there exist social welfare functions satisfying Hammond Equity plus DWP by Theorem 1, but there does not exist any social welfare function satisfying Strong Equity plus Monotonicity. So, we can say that the axiom set of Strong Equity plus Monotonicity places a *stronger restriction* on domain sets Y than the axiom set of Hammond Equity plus DWP, for the existence of social welfare functions.

Acknowledgments We thank the Editor and two anonymous referees of this journal for their comments and suggestions.

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[9] For a social welfare function $W : X \to \mathbb{R}$, the Strong Equity axiom can be stated as follows. If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$, while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $W(x) > W(y)$. That is, under the same hypothesis as in Hammond Equity, it asserts the stronger implication that $W(x) > W(y)$.

[10] This is stated as property (S1) in Diamond (1965, p. 172).

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